Eidgenössische
Technische Hochschule
Zürich

Ecole polytechnique fédérale de Zurich
Politecnico federale di Zurigo
Federal Institute of Technology at Zurich

Departement of Computer Science
Markus Püschel, David Steurer

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Algorithms & Data Structures — Homework 2 — HS 18

Exercise Class (Room & TA): ____________________

Submitted by: ____________________

Peer Feedback by: ____________________

Points: _____

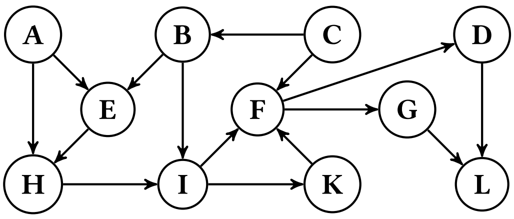
Exercise 2.1 *Number of Edges and Subgraphs.*

Answer the following questions and justify your answers with a brief explanation.

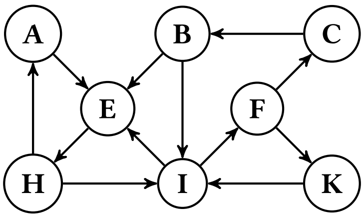
1. How many edges does an undirected graph of n vertices maximally contain? How many edges does a directed graph of n vertices maximally contain? (In both cases, assume that the graph does not contain loops.)
2. What is the maximum number of edges in an undirected k-partite graph with $n = \sum_{i=1}^{k} u_k$ vertices, where $u_i > 0$ is the number of vertices in the i-th subset of this partition?
3. Given an undirected clique G of size n, where n is an odd prime number. How many pairwise edge-disjoint simple cycles (i.e. cycles that use every vertex at most once) of length n does G contain?

Exercise 2.2 *Topological Sorting* **(1 point)**.

1. How many topological orders does the following graph contain? List all topological orders.



2. Consider now the following graph $G = (V, E)$ and find a set $E' \subset E$ of minimum cardinality, such that $G' = (V, E \setminus E')$ can be topologically sorted. Justify your answer (i.e., why is it necessary to remove at least $|E'|$ edges?).



3. What is the maximum number of edges in a directed graph that can be topologically sorted? Formulate your claim for every positive integer n and prove your claim by using induction.

   Hint: *For each $n \in \{1, 2, 3, 4\}$ construct a graph with n vertices that is topologically sortable and contains the maximum number of edges. Then use these observations to derive your claim.*)

Exercise 2.3 *Number of Edges and Connected Components.*

1. Prove via mathematical induction that a connected graph with $n > 0$ vertices has at least $n - 1$ edges.
2. Prove that a graph G with n vertices and m connected components has at least $n - m$ edges.

   Recall that an undirected acyclic graph is called a *forest*. It is easy to see that a graph G is a forest iff each connected component of G a is a tree.
3. Prove that a forest G with n vertices and m connected components has $n - m$ edges.
4. Prove that if a graph G with n vertices and m connected components has $n - m$ edges, then G is a forest.

Exercise 2.4 *Hamiltonian paths in directed acyclic graphs* **(2 points)**.

A *Hamiltonian path* in a (directed or undirected) graph G is a path in G that visits each vertex of G exactly once. It is known that a problem of finding a Hamiltonian path in a graph is **NP**-hard, which means that it is highly unlikely that this problem can be solved in polynomial time. However, for special types of graphs it is possible to solve this problem efficiently.

For directed acyclic graphs one can find a Hamiltonian path using topological sorting. To show this, answer the following questions about topological orderings and Hamiltonian paths:

1. Let G be a directed acyclic graph which has a Hamiltonian path. What is the relationship between the set of Hamiltonian paths and the set of topological orderings of G? What are the sizes of these sets?
2. Let G be a directed acyclic graph with no Hamiltonian paths. Can G have a unique topological ordering?

Exercise 2.5 *Eulerian tours.*

An *Eulerian tour* is a closed walk (Zyklus) that visits every edge exactly once.

In this exercise, we ask you to prove that a connected graph contains an Eulerian tour if and only if it does not contain a vertex of odd degree.

1. Prove that if a connected graph G contains an Eulerian tour, then G does not contain a vertex of odd degree.
2. Prove that every connected graph without vertices of odd degree contains a Eulerian tour. Use mathematical induction on the number of edges.

   Hint: *Use the fact that every non-trivial connected graph without vertices of odd degree contains a cycle. Notice that this fact is a direct consequence of the fact that every non-trivial acyclic graph contains a leaf (which you proved in the previous exercise sheet).*

Submission: On Monday, 8.10.2018, hand in your solution to your TA *before* the exercise class starts.